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A simple derivation of a mean and variance in a truncated inverse sampling problem

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Abstract

Let Z be a negative binomial random variable with parameters m, p, and let $M \geqq m$ be a positive integer. In Chang (2001), mathematical induction was used to prove an identity for $E[\min(Z, M)]$. We give a simpler derivation of the the mean and also derive $Var[\min(Z, M)]$.

Keywords: Inverse sampling, truncated inverse sampling, mathematical induction.

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Suppose that in each time period an event independently occurs with probability p. Let Z be the time of the mth event; let M, ($M \geqq m$), be a specified integer; and let $S = Z - \min(Z, M)$ be the amount of time that is saved when one uses the truncation time M while waiting for m events to occur. If we let N denote the number of events by time M, then given N, the random variable S represents the amount of time needed for $(m-N)^+$ events. Hence, with $q = 1 - p$,

$$E(S|N) = \frac{(m-N)^+}{p} \quad \text{and} \quad Var(S|N) = \frac{q(m-N)^+}{p^2} \ .$$

Consequently,

$$E(S) = \sum_{j=0}^{m-1} \frac{m-j}{p} \binom{M}{j} p^j q^{M-j} \qquad (1)$$

showing that

$$E[\min(Z,M)] = E(Z) - E(S) = \frac{m}{p} - \sum_{j=0}^{m-1} \frac{m-j}{p} \binom{M}{j} p^j q^{M-j} . \qquad (2)$$

The conditional variance formula (see Ross 2000) gives

$$\begin{aligned} Var(S) &= Var[E(S|N)] + E[Var(S|N)] \\ &= E\{[E(S|N)]^2\} - [E(S)]^2 + \frac{q}{p} E(S) \\ &= \sum_{j=0}^{m-1} \frac{(m-j)^2}{p^2} \binom{M}{j} p^j q^{M-j} - [E(S)]^2 + \frac{q}{p} E(S) . \end{aligned} \qquad (3)$$

Now, given $N < m$, $Z = M + S$. Hence,

$$\begin{aligned} E(SZ) &= \sum_{i=0}^{m-1} E\,[S(M+S)\,|N=j] \binom{M}{j} p^j q^{M-j} \\ &= M \cdot E(S) + \sum_{j=0}^{m-1} \left[\frac{(m-j)q}{p^2} + \frac{(m-j)^2}{p^2} \right] \binom{M}{j} p^j q^{M-j} \\ &= (M + \frac{q}{p}) E(S) + \sum_{j=0}^{m-1} \frac{(m-j)^2}{p^2} \binom{M}{j} p^j q^{M-j} . \end{aligned}$$

Consequently,

$$Cov(S, Z) = (M + \frac{q}{p} - \frac{m}{p})E(S) + \sum_{j=0}^{m-1} \frac{(m-j)^2}{p^2} \binom{M}{j} p^j q^{M-j}. \qquad (4)$$

From Equations (3) and (4) we obtain

$$Var[\min(Z, M)]$$

$$= Var(Z) + Var(S) - 2Cov(S, Z)$$

$$= \frac{mq}{p^2} - \frac{1}{p^2}\sum_{j=0}^{m-1}(m-j)^2 \binom{M}{j} p^j q^{M-j} - [E(S)]^2 - (2M + \frac{q}{p} - \frac{2m}{p})E(S) \qquad (5)$$

where $E(S)$ is given by Equation (1).

Remarks：

a) Equation (2) was previously derived in Chang (2001) by mathematical induction. The equivalence between Equation (2) and the result of Theorem 2.2 in Chang (2001) can be seen by interchanging the order of summation of the result given in that paper.

b) The usefulness of the identities (2) and (5) is that one only needs sum to $m - 1$, which is typically much smaller than M.

References

Chang, K-C (2001). An inductive proof for a closed form formula in truncated inverse sampling, *Journal of Propagations in Probability and Statistics*, **2,** 1, 117-122.

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