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An Alternative Proof for a Continuity Property of Positive Definite Matrices

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ABSTRACT This note concerns an article by Hui-Kuang Hsieh, which appeared in the August 2004 issue of this journal. The note provides an alternative method for proving the main proposition in that article without relying on a property of continuous functions used by the author. The proposed method is also easy to use from the computational point of view. In addition, an error is pointed out in the statement of Hsieh's proposition.

Keywords Hsieh's proposition; Quadratic form; Spectral decomposition theorem.

1. Hsieh's Proposition

Hsieh [3] stated and proved the following proposition:

Proposition Let $\mathbf{\Sigma}$ be a $p \times p$ positive definite and symmetric matrix, and $\mathbf{A}$ be any $p \times p$ matrix. Then there exists a positive number t_0 such that $\mathbf{M}(t) = \mathbf{\Sigma} - t\mathbf{A}$ is positive definite for all t with $|t| < t_0$.

In proving this proposition, the author invoked a certain result stated in Rao ([5], page 36) giving the necessary and sufficient condition for $\mathbf{M}(t)$ to be positive definite. This result, however, requires that $\mathbf{M}(t)$ be symmetric (see also Harville [2], Theorem 14.9.5, page 247, and Graybill [1], Theorem 12.2.2, page 396). Consequently, $\mathbf{A}$ must also be symmetric. The proposition, however, states that $\mathbf{A}$ is "any $p \times p$ matrix". Hence, Hsieh's proof is not

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valid unless $\mathbf{A}$ is considered to be symmetric. The numerical example given by Hsieh uses a symmetric matrix value for $\mathbf{A}$.

2. An Alternative Proof for Hsieh's Proposition

Consider again the matrix $\mathbf{M}(t) = \boldsymbol{\Sigma} - t\mathbf{A}$, where $\mathbf{A}$ is a symmetric matrix. It is easy to see that $\mathbf{M}(t)$ is positive definite if and only if the matrix

$$\mathbf{L}(t) = \mathbf{I}_p - t\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}} \tag{2.1}$$

is positive definite since $\mathbf{L}(t) = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{M}(t)\boldsymbol{\Sigma}^{-\frac{1}{2}}$, where $\boldsymbol{\Sigma}^{-\frac{1}{2}}$ is obtained by first writing $\boldsymbol{\Sigma}$ as $\boldsymbol{\Sigma} = \mathbf{P}\boldsymbol{\Delta}\mathbf{P}'$ using the spectral decomposition theorem, where $\boldsymbol{\Delta}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{\Sigma}$, which are positive, and $\mathbf{P}$ is an orthogonal matrix whose columns are the corresponding eigenvectors of $\boldsymbol{\Sigma}$ (see, for example, Graybill [1], Theorem 3.4.4, page 48). The matrix $\boldsymbol{\Sigma}^{-\frac{1}{2}}$ is then defined as $\boldsymbol{\Sigma}^{-\frac{1}{2}} = \mathbf{P}\boldsymbol{\Delta}^{-\frac{1}{2}}\mathbf{P}'$, where $\boldsymbol{\Delta}^{-\frac{1}{2}}$ is a diagonal matrix whose diagonal elements are the positive square roots of the reciprocals of the diagonal elements of $\boldsymbol{\Delta}$. Since the matrix $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}}$ is symmetric, we can use the spectral decomposition theorem again to write

$$\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}', \tag{2.2}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}}$ and $\mathbf{Q}$ is an orthogonal matrix whose columns are the corresponding eigenvectors. Note that the eigenvalues of $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}}$ are the same as those of $\mathbf{A}\boldsymbol{\Sigma}^{-1}$. This is true by Theorem 9 in Magnus and Neudecker [4] (page 14) and the fact that $\mathbf{A}$ and $\boldsymbol{\Sigma}$ are square matrices of the same order. From (2.1) and (2.2) we then have

$$\mathbf{L}(t) = \mathbf{Q}(\mathbf{I}_p - t\boldsymbol{\Lambda})\mathbf{Q}'. \tag{2.3}$$

Hence, $\mathbf{L}(t)$ is positive definite if and only if

$$1 - t\lambda_i > 0, \quad i = 1, 2, ..., p, \tag{2.4}$$

where λ_i is the i^{th} eigenvalue of $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}}$ (or $\mathbf{A}\boldsymbol{\Sigma}^{-1}$). But,

$$t\lambda_i \leq |t||\lambda_i| \leq |t| \max_i |\lambda_i|, \quad i = 1, 2, ..., p. \tag{2.5}$$

Thus in order to satisfy all p inequalities in (2.4), it is sufficient to choose t such that

$$|t| < \frac{1}{\max_i |\lambda_i|}. \tag{2.6}$$

Note that $\max_i |\lambda_i| > 0$ since at least one eigenvalue of $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}}$ must be different from zero. If not, then $\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{A}\boldsymbol{\Sigma}^{-\frac{1}{2}} = \mathbf{0}$ resulting in $\mathbf{A}$ being a zero matrix. In this case, $\mathbf{M}(t) = \boldsymbol{\Sigma}$, which is positive definite for all t and, therefore, there is nothing to prove. The value of t_0 in Hsieh's proposition can then be chosen equal to

$$t_0 = \frac{1}{\max_i |\lambda_i|}. \tag{2.7}$$

3. Advantage of the Alternative Method

The proposed method for proving Hsieh's proposition uses only simple matrix results and does not utilize any calculus techniques, as was the case in Hsieh's proof, which is structured around the continuity property of polynomials. However, the main advantage of the propsed method is in providing an actual value for the upper bound t_0 as given in (2.7). Hsieh's proof shows the existence of such a value, but the actual determination of t_0 (using Hsieh's proof) requires finding solutions to p inequalities involving polynomials of degrees ranging from 1 to p. This can be very complicated from the computational point of view, particularly when p is large. The value of t_0 in (2.7) can be easily found, even for large p, since it only requires computing the eigenvalues of $\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{A}\mathbf{\Sigma}^{-\frac{1}{2}}$ (or $\mathbf{A}\mathbf{\Sigma}^{-1}$). This can be conveniently done using, for example, PROC IML in SAS [6], or any other comparable computer package.

Knowledge of the value of t_0 is important since, after all, the main purpose of Hsieh's proposition is to find an interval for t over which $\phi(t)$ exists, where $\phi(t)$ is the moment generating function of the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ with $\mathbf{x}$ being a normally distributed random vector. The value of t_0 in (2.7) clearly shows that the domain of definition of $\phi(t)$ depends on the eigenvalues of the matrix $\mathbf{A}\mathbf{\Sigma}^{-1}$.

4. Hsieh's Numerical Example

Hsieh considered the following values for $\mathbf{A}$ and $\mathbf{\Sigma}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \qquad \mathbf{\Sigma} = \begin{bmatrix} 3 & 5 & 1 \\ 5 & 13 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}\mathbf{\Sigma}^{-1}$ are -4, 0.381966, 2.618034. Hence, $\max_i |\lambda_i| = 4$. Thus, from (2.7), $t_0 = 0.25$. This agrees with the value found by Hsieh after solving three inequalities involving polynomials of degrees 1, 2, and 3. Even in this simple example, finding the solution to the third inequality (for a polynomial of degree 3) was somewhat complicated.

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